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ANALYSIS OF THE STABILITY OF STOCHASTIC VISCOELASTIC SYSTEMS†

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The RMS stability of viscoelastic systems, subject to parametric loads, with respect to a perturbation of the initial conditions is considered. These loads, as well as the external damping characteristics, are assumed to be random, steady-state, broadband processes. An expansion of the required displacements in a series in the eigenfunctions of the elastic system is used to solve the functional equations which describe the motion of such systems. The integro-differential equations in generalized displacements are solved using the method of stochastic averaging (an asymptotic method) under the assumption that the measure of the relaxation of the material and the mean value of the external damping are small quantities compared with unity and that the random fluctuations of the processes under consideration are small in the mean square. It is shown that the critical value of the system parameter may turn out to be least when the form of the perturbations of the initial conditions differs from the first form of the natural vibrations. © 1997 Elsevier Science Ltd. All rights reserved.

It is often assumed, when investigating the stability of "classical" elements of elastic structures, such as, for example, a rod which is hinged at the ends and is acted upon by a longitudinal force which is constant along its length, or a rectangular plate which is freely supported along its edges and is acted upon by uniformly distributed compressive loads applied at the edges in the plane of the plate, etc., that the form of the deflection of the element is identical to the form of the first mode of its natural vibrations. The treatment of a distributed system as a system with an infinite number of degrees of freedom is thereby replaced by an analysis of the behaviour of a far simpler model, a system with one degree of freedom.

As a rule, the conditions for the stability of the equilibrium position of the system obtained using such a model are identical to the analogous condition for a distributed system. Not infrequently, a similar approach is extended to systems made of a material with more complex rheological properties, in particular, viscoelastic properties. However, in certain situations, this approach may lead to conclusions which are qualitatively incorrect. The example treated in this paper confirms this.

It was shown, in an analysis of the stability of the zeroth solution of a stochastic system of integro-differential equations [1], using the example of a viscoelastic rod, that perturbations of the initial conditions which differ in form from a single sinusoidal half-wave can, from the point of view of the stability, turn out to be more dangerous than those perturbations which have the form of a single sinusoidal half-wave. But the stability conditions are found using the direct Lyapunov method and are therefore sufficient but not necessary. The question therefore remains open as to whether the result which is obtained is just a consequence of the method used to solve the problem or whether the cause is of a more profound nature. In order to elucidate the cause of the difference which has been noted, it is necessary to find the conditions for the stability of a stochastic system which should be sufficient and necessary.

Since, at the present time, there are no exact methods for solving stochastic equations with parameters which are broadband, steady-state processes, we shall use the method of stochastic averaging [2–8] for this purpose. This method is a development of the asymptotic method [9–11] for solving differential and integro-differential equations.

Of the papers which deal with the problem of the stability of stochastic mechanical systems, we note the investigations of the stability of a viscoelastic rod supported by a hinge at the ends and which is acted upon by a random longitudinal force in the form of white noise [12, 13] where sufficient conditions for stability were obtained under very general assumptions regarding the rod material relaxation kernel and taking ageing into account and ignoring it. It was shown that, when there is no external damping, the rod turns out to be asymptotically stable in the mean square if the stability condition is satisfied when the initial conditions are perturbed, the form of which is identical to a single sinusoidal half-wave. An analysis of elastic, linear and non-linear systems has been given using the method of stochastic

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averaging and Lyapunov exponents [14, 15]. The application of the method of the maximum Lyapunov exponent was considered in [16] when investigating elastic systems with two degrees of freedom which are perturbed by white noise.

1. FORMULATION OF THE PROBLEM

The motion of a viscoelastic system is described by an equation which can be represented in operator form as

$$\frac{\partial^2 u}{\partial t^2}+\mathbf{A}\frac{\partial u}{\partial t}+(1-\mathbf{R})\mathbf{B}u-\mathbf{C}u=0 \tag{1.1}$$

$$\mathbf{R}u=\int_{-\infty}^{t} R(t-\tau)u(\mathbf{x},\tau)d\tau, \qquad 0\leqslant\int_0^{\infty} R(\theta)d\theta\ll 1$$

where $u(t, \mathbf{x})$ is the displacement of the system, $\mathbf{x}$ is a vector of the spatial coordinates, $\mathbf{R}$ is an integral operator, $R(t-\tau)$ is the relaxation kernel of the material, and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are partial differential operators with respect to the spatial coordinates.

The solution of Eq. (1.1) must satisfy the corresponding initial and boundary conditions.

The terms $\mathbf{A}\partial u/\partial t$, $\mathbf{B}u$, $\mathbf{C}u$ take account of the external damping, the rigidity of the elastic system and the action of the load parameters, respectively.

If $\varphi_i(\mathbf{x})$ are the eigenfunctions of the boundary-value problem $\mathbf{B}\varphi = \omega^2\varphi$, then u can be expanded in a Fourier series in these functions

$$u(\mathbf{x},t)=\sum_{i=1}^{\infty} f_i(t)\varphi_i(\mathbf{x}) \tag{1.2}$$

The relations

$$\int_V \varphi_i\varphi_j dV=\delta_{ij}, \quad \int_V (\mathbf{B}\varphi_i)\varphi_j dV=\omega_i^2\delta_{ij} \tag{1.3}$$

hold for the functions $\varphi_i(\mathbf{x})$ where δ_{ij} is the Kronecker delta, ω_i is the frequency of natural oscillations of the system and V is the volume of the system.

If the operator $\mathbf{A}$ takes account of external friction only, we can put

$$\int_V (\mathbf{A}\varphi_i)\varphi_j dV=2\varepsilon^*\delta_{ij} \tag{1.4}$$

Here ε^* is a characteristic of the external damping.

In certain cases, it can also be assumed that the equality

$$\int_V (\mathbf{C}\varphi_i)\varphi_j dV=\omega_i^2\alpha_i\delta_{ij} \tag{1.5}$$

is satisfied, where α_i is a dimensionless parameter which characterizes the parametric load.

Equality (1.5) holds, for example, in the case of a rod of constant cross-section which is supported by hinges at the ends and is acted upon by longitudinal forces applied at its ends. The same can also be said about a rectangular plate which is supported by hinges along all its edges and is subject to uniformly distributed loads in the plane of the plate and parallel to its edges, a circular cylindrical shell under uniform axial compression, and so on.

Taking account of relations (1.3)–(1.5), from Eq. (1.1) we obtain the equations for the generalized displacements f_i

$$\ddot{f}_i+2\varepsilon^*\dot{f}_i+(1-\mathbf{R})\omega_i^2 f_i-\omega_i^2\alpha_i f_i=0 \tag{1.6}$$

(a dot over a symbol denotes a derivative with respect to time t).

2. SOLUTION OF THE STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

Next, we shall assume that the external damping characteristics ε^* and the parametric load characteristics α_i are uncorrelated steady processes

$$\varepsilon^*(t)=\varepsilon_0^*+\varepsilon_1^*(t),\quad \alpha_i=\alpha_{i0}+\alpha_{i1}(t)$$

$$\varepsilon_0^*=\langle\varepsilon^*(t)\rangle=\text{const},\quad \alpha_{i0}=\langle\alpha_i(t)\rangle=\text{const},\quad \alpha_{i0}<1$$

and that the parameter ε_0/ω_1 is small compared with unity and the functions $\varepsilon_1^*(t)/\omega$ and α_{i1} are small in the mean square.

Here and henceforth the operation of mathematical expectation is denoted by angle brackets.

We now represent Eq. (1.6) in the form of the system

$$\dot f_i=\psi_i$$
$$\dot\psi_i=-2\varepsilon^*\psi_i-\Omega_i^2 f_i+\omega_i^2\int_{\infty}^{t}R(t-\tau)f_i(\tau)d\tau+\omega_i^2\alpha_{i1}f_i \tag{2.1}$$

where $\Omega_i^2=\omega_i^2(1-\alpha_{i0})$.

A solution of Eqs (2.1) is sought in the form [2–9]

$$f_i=A_i\sin\theta_i,\quad \psi_i=\Omega_iA_i\cos\theta_i,\quad \theta_i=\Omega_it+\nu_i \tag{2.2}$$

Making the usual transformations in the case of the asymptotic method, we express the derivatives of the functions $\dot A$ and $\dot\nu$. Averaging the non-fluctuating terms over a period of the oscillations $T=2\pi/\omega_i$ and introducing the dimensionless time $t_1=\omega_1t$, we arrive at the equations

$$A_i'=F_1(A_i,\nu_i)+g_{11}\eta_1+g_{12}\eta_2,\qquad \nu_i'=F_2(A_i,\nu_i)+g_{21}\eta_1+g_{22}\eta_2 \tag{2.3}$$

Here

$$F_1(A_i,\nu_i)=-\varepsilon_0A_i+\mu_i^2zA_i,\quad F_2(A_i,\nu_i)=-\mu_i^2x,\quad \mu_i^2=\omega_i^2/(\omega_1\Omega_i)$$

$$z=-\frac{1}{2}\int_0^{\infty}R(y)\sin\Omega_iydy,\quad x=\frac{1}{2}\int_0^{\infty}R(y)\cos\Omega_iydy$$

$$g_{11}=-2A_i\cos^2\theta_i,\quad g_{12}=-\frac{1}{2}\mu_i^2A_i\sin2\theta_i,\quad g_{21}=\sin2\theta_i$$

$$g_{22}=-\mu_i^2\sin^2\theta_i,\quad \eta_1=\varepsilon_1,\quad \eta_2=\alpha_{i1},\quad \varepsilon_0=\varepsilon_0^*/\omega_1,\quad \varepsilon_1=\varepsilon_1^*/\omega_1$$

A derivative with respect to the time t_1 is denoted by a prime.

For convenience, we shall subsequently use the notation t instead of t_1.

We next average the fluctuating terms, as a result of which the random processes A_i are Markov diffusional processes

$$A_i'=a_i+b_i\xi_i \tag{2.4}$$

where ξ_i is the equivalent steady white noise with a mathematical expectation of zero and a correlation function $K(\tau)=\delta(\tau)$, where $\delta(\tau)$ is the δ-function.

We shall find the functions a_i, b_i in the manner suggested in [2–4].

After all the above algebra, we obtain

$$a_i=F(A_i)+A_i\int_{-\infty}^{0}\left[1+\frac{3}{2}\cos\left(2\frac{\Omega_i}{\omega_1}\upsilon\right)\right]K_{11}(\upsilon)d\upsilon+\frac{3}{8}\mu_i^4A_i\int_{-\infty}^{0}\cos\left(2\frac{\Omega_i}{\omega_1}\upsilon\right)K_{22}(\upsilon)d\upsilon\equiv c_iA_i$$

$$b_i^2=\left\{\int_{-\infty}^{\infty}\left[1+\frac{1}{2}\cos\left(2\frac{\Omega_i}{\omega_1}\upsilon\right)\right]K_{11}(\upsilon)d\upsilon+\frac{1}{8}\mu_i^4\int_{-\infty}^{\infty}\cos\left(2\frac{\Omega_i}{\omega_1}\upsilon\right)K_{22}(\upsilon)d\upsilon\right\}A_i^2\equiv\beta_i^2A_i^2$$

Here, $K_{11}(v)$, $K_{22}(v)$ are the correlation functions for the processes ε_1 and α_{i1} respectively.

It is seen from the expressions for a_i, b^2_i that the amplitude A_i is independent of the phase of the oscillations v_i. It will not have any effect on the estimation of the stability, and the equation from which the function v_i is determined can therefore be excluded from further discussion.

Equation (2.4) can therefore be written in the form

$$A_i' = c_i A_i + \beta_i A_i \xi_i$$

Using Ito's differentiation formula (or the Fokker–Planck–Kolmogorov equation), we obtain an equation in the dispersion $\langle A^2_i \rangle$

$$\langle A_i^2 \rangle' = 2c_i \langle A_i^2 \rangle + \beta_i^2 \langle A_i^2 \rangle \equiv \lambda_i \langle A_i^2 \rangle \tag{2.5}$$

3. STABILITY OF A VISCOELASTIC SYSTEM

We introduce the norm in the space of the functions $u(\mathbf{x}, t)$

$$\| u(t) \|^2 = \int_V u^2(\mathbf{x}, t) dV$$

We shall say that the equilibrium position of the system ($u \equiv 0$) is stable in the mean square with respect to a perturbation of the initial conditions if, for any $\Delta > 0$ which may be as small as desired, a $\delta(\Delta) > 0$ is found such that the inequality $\langle \| u(t) \|^2 \rangle < \Delta$ follows, which is satisfied at any instant of time $t > 0$, from the condition $\langle \| u(0) \|^2 \rangle < \delta$ which holds for the initial instant of time $t = 0$.

The equilibrium position of the system is said to be asymptotically stable in the mean square if the previous condition is satisfied and, in addition, a $\delta > 0$ is found such that the equality $\lim_{t \to \infty} \langle \| u(t) \|^2 \rangle = 0$ holds when $\langle \| u(0) \|^2 \rangle < \delta$.

Taking account of expansion (1.2) and bearing in mind the first equality of (2.2), we obtain the inequality

$$\langle \| u(t) \|^2 \rangle = \sum_{i=1}^{\infty} \langle f_i^2(t) \rangle \leq \sum_{i=1}^{\infty} \langle A_i^2(t) \rangle$$

for the dispersion $\langle \| u(t) \|^2 \rangle$.

The dispersion $\langle A_i^2(t) \rangle$ turns out to be a monotonically decaying function if the condition

$$\lambda_i = 2c_i + \beta_i^2 < 0$$

is satisfied.

Substituting the expressions for c_i and β_i^2, we finally write

$$\lambda_i = \int_0^{\infty} \left[2 + 2\cos\left(2\frac{\Omega_i}{\omega_1} v \right) \right] K_{11}(v) dv + \frac{\mu_i^4}{2} \int_0^{\infty} \cos\left(2\frac{\Omega_i}{\omega_1} v \right) K_{22}(v) dv -$$

$$-\varepsilon_0 - \frac{\mu_i^2}{2} \int_0^{\infty} R(y) \sin\left(\frac{\Omega_i}{\omega_1} y \right) dy < 0 \tag{3.1}$$

This condition can be considered as the condition for the asymptotic stability in the mean square in the case of a system with a perturbation of the initial conditions, the form of which is identical to the ith mode of the natural oscillations $\varphi_i(\mathbf{x})$.

It is obvious that, in the case of a perturbation of the initial conditions of arbitrary form, the system will be asymptotically stable in the mean square if condition (3.1) is satisfied for any i.

4. EXAMPLE

Consider a rectilinear rod of constant cross-section hinged at the ends and acted upon by a longitudinal force $F(t)$. In this case, Eq. (1.1) is written in the form

$$\frac{\partial^2 w}{\partial t^2} + 2\varepsilon^* \frac{\partial w}{\partial t} + \frac{EI}{m}(1-\mathrm{R})\frac{\partial^4 w}{\partial x^4} - \frac{F}{m}\frac{\partial^2 w}{\partial x^2} = 0$$

Here, w is the deflection of the rod, x is a coordinate measured along the axis of the rod from one of its ends, EI is the flexural rigidity of the rod and m is its mass per unit length.

The boundary conditions when $x = 0$ and $x = l$ (l is the length of the rod) have the form $w = \partial^2 w/\partial x^2 = 0$. The functions $\varphi_i(x)$, the natural frequencies ω_i, the quantities μ^2_i, α_{i0} and the random function α_{i1} are defined in this case by the expressions

$$\varphi_i(x) = \sqrt{\frac{2}{l}}\sin\frac{i\pi}{l}x, \quad \omega_i^2 = \frac{i^4\pi^4}{l^4}\frac{EI}{m}, \quad \mu_i^2 = \frac{i^2}{\sqrt{1-\alpha_{i0}}}$$

$$\alpha_{i0} = \frac{\alpha_{10}}{i^2}, \quad \alpha_{i1} = \frac{\alpha_{11}(t)}{i^2}$$

We shall assume that the kernel for the relaxation of the rod material and the correlation functions of the steady-state processes ε_1 and α_{11} are exponential functions and have the form

$$R(t-\tau) = Me^{-\chi(t-\tau)}, \quad K_{11}(\tau) = s^2 e^{-\rho|\tau|}, \quad K_{22}(\tau) = \sigma^2 e^{-\gamma|\tau|}/i^4$$

where M, χ, s, ρ, σ, γ are positive constants (dimensionless quantities in the same way as t and τ).

Then

$$z = -\frac{1}{2}M\frac{\Omega_i}{\omega_1}\left(\frac{\Omega_i^2}{\omega_1^2} + \chi^2\right)^{-1}$$

Finally, the quantity λ_i (3.1) is defined by the expression

$$\lambda_i = P_i + Q_i s^2 \tag{4.1}$$

$$P_i = -2\varepsilon_0 - M\left(1 + \frac{\chi^2}{i^4} - \alpha_{i0}\right)^{-1} + \frac{\sigma^2\gamma}{1-\alpha_{i0}}[\gamma^2 + 4i^4(1-\alpha_{i0})]^{-1}$$

$$Q_i = \frac{8}{\rho}\frac{\rho^2 + 2i^4(1-\alpha_{i0})}{\rho^2 + 4i^4(1-\alpha_{i0})}$$

It was noted in [1] that, when $\alpha_{i1} \equiv 0$, a viscoelastic rod is stable in the mean square if the following condition, obtained as $i \to \infty$, is satisfied

$$2\langle|\varepsilon_1|\rangle < \varepsilon_0 + M/2 \tag{4.2}$$

Assuming that ε_1 is a Gaussian process with zero mathematical expectation and variance s^2, we find that $\langle|\varepsilon_1|\rangle = \sqrt{(2/\pi)}s$, as a result of which inequality (4.2) takes the form

$$s < \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\varepsilon_0 + \frac{M}{2}\right) \tag{4.3}$$

For comparison, we return to inequality (3.1). Taking account of expression (4.1) for λ_i, we express the variance s^2 from it

$$s^2 < S^2, \quad S^2 = -P_i/Q_i \tag{4.4}$$

In the case of the example in Table 1, some of the values of S^2 when $\alpha_{i0} = 0.5$; $\varepsilon_0 = 0.1$; $M = 0.01$; $\chi = 0.1$; $\sigma^2 = 0$ are given in the second and third columns.

An analysis of the data suggests that the quantity S^2 can take a minimum (critical) value when $i \neq 1$. This means that, from the point of view of the mean-square stability, modes of initial deflection of the rod which differ from the first mode of natural oscillations may turn out to be more dangerous. In this sense, the sufficient condition for mean-square stability (4.3) which is obtained when $i \to \infty$, is confirmed by the inequality (4.4), at least for certain combinations of the parameters.

Table 1

i	$S^2 \times 10^5$ when $\rho = 0.1$	$S^2 \times 10^5$ when $\rho = 0.01$	$\beta_1^2 \times 10^5$
1	546	54.9	10 943
2	528	52.9	10 438
∞	525	52.5	10 500

5. STEADY PROCESSES OF THE WHITE-NOISE TYPE

For comparison, we will now consider the stability of a rod when the functions $\varepsilon_1(t)$ and $\alpha_{11}(t)$ are steady Gaussian white noises.

In the case of an exponential relaxation kernel $R(t-\tau) = M \exp[-\chi(t-\tau)]$, the system of integro-differential equations (2.1) can be replaced, using the substitution

$$X_i(t) = \int_{-\infty}^{t} M e^{-\chi(t-\tau)} f_i(\tau) d\tau$$

by the following system of first-order differential equations [17]

$$\begin{aligned} \dot{f} &= \psi_i \\ \dot{\psi}_i &= -2\varepsilon^* \psi_i - \Omega_i^2 f_i + \omega_i^2 X_i + \omega_i^2 \alpha_{i1} f_i \\ \dot{X}_i &= \omega_1 (M f_i - \chi X_i) \end{aligned} \tag{5.1}$$

Using Ito's formula, we obtain from this a closed system of differential equations in the statistical moments $\langle f_i^2 \rangle$, $\langle f_i \psi_i \rangle$, $\langle f_i X_i \rangle$, $\langle \psi_i^2 \rangle$, $\langle \psi_i X_i \rangle$, $\langle X_i^2 \rangle$ [17] with a constant coefficient matrix.

From an analysis of the characteristic equation we find the condition for the asymptotic stability in the mean square of the zeroth solution of Eqs (5.1).

If the white noises are understood in Ito's sense, this condition can be written as follows:

$$\begin{aligned} &4(\beta_1^2 - \varepsilon_0)\left(1 - \alpha_{i0} + \xi \frac{\omega_1^2}{\omega_i^2}\right) + \left(\frac{\omega_i^2}{\omega_1^2} + \frac{\xi}{\delta}\right)\beta_{2i}^2 < 2M \\ &\xi = \chi(2\varepsilon_0 + \chi), \quad \delta = 1 - \alpha_{i0} - M/\chi \end{aligned} \tag{5.2}$$

where β_1^2, β_{2i}^2 are the intensities of the uncorrelated white noises β_1^2 and $\alpha_{i1}(t)$.

Inequality (5.2) holds when the conditions $\varepsilon_0 > \beta_1^2$ and $\delta > 0$ are satisfied. The latter is the condition for asymptotic stability of a deterministic system (when $\varepsilon \equiv \varepsilon_0$, $\alpha \equiv \alpha_{i0}$). From relation (5.2) when $\beta_{2i}^2 = 0$ in the case of a rod we obtain

$$\beta_1^2 < \varepsilon_0 + \frac{M}{2}\left(1 - \frac{\alpha_{10}}{i^2} + \frac{\xi}{i^4}\right)^{-1}$$

The values of the right-hand side of the inequality for the same values of the parameters ε, χ, M, α_{10} for which the values of S^2 were previously found are shown in the fourth column of Table 1.

It can be seen that, as in the case of white noise, forms of perturbations of the initial conditions which involve a number of half-waves greater than unity may turn out to be more dangerous. It is interesting that the relation between the parameter β_1^2 and the number of half-waves can be non-monotone since the values of β_1^2 when $i = 2$ turned out to be smaller than when $i = 1$ and $i \to \infty$.

It should be noted that the replacement of white noises in Ito's sense by the white noises in the treatment of Stratonovich does not change the qualitative conclusions.

We also note that the results found using the asymptotic method only hold for small values of the quantities M/χ, s and σ while no such constraints, apart from the constraint that $M/\chi < 1$, are imposed on M/χ, β_i and β_{21}. The constraint $M/\chi < 1$ is due to the boundedness of the viscosity of the system material.

It is necessary to bear in mind that inertial forces resulting from shear deformations and the rotation of cross-sections were not taken into account in writing the equation of motion of the rod. The values of the quantity S^2, found using expression (4.3) for large values of i, are therefore approximate. Moreover, it should be kept in mind that, while the quantitative results are refined when account is taken of shear and rotational inertia, this does not change the qualitative conclusion that the critical value of the parameter s^2 may correspond to a sinusoidal form of the initial deformation with a number of half-waves greater than unity.

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